Delta method for variance estimation

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Variance of a function of a single random variable

- X is a random variable with expected value μ_X and variance σ_X^2.
- A new variable Y is defined by $f(X)$ where derivatives of $f(X)$ with respect to X exist (up to some order).
- The function $f(X)$ can be approximated by a 1st order Taylor series, where X is evaluated at μ_X:

$$\hat{f}(X) \approx f(\mu_X) + f'(\mu_X)(X - \mu_X)$$

- The variance of $f(X)$ can be approximated by taking the variance of $\hat{f}(X)$.

$$\begin{aligned} \text{Var}[f(X)] &\approx \text{Var}[\hat{f}(X)] \\ &= \text{Var}[f'(\mu_X)(X - \mu_X)] \\ &= (f'(\mu_X))^2 \sigma_X^2. \end{aligned} \tag{1}$$

- Example: Suppose $Y = \sqrt{X}$. Then $f'(X) = 1/2 \, \frac{1}{\sqrt{X}}$.

$$\text{Var}[\sqrt{X}] \approx \left(\frac{1}{2\sqrt{\mu_X}} \right)^2 \sigma_X^2 = \frac{1}{4\mu_X} \sigma_X^2$$

Variance of a function of several random variables

- X and Y are random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}.
- A new variable Z is defined as a function of X and Y, Z=$f(X, Y)$. Approximate $f(X, Y)$ again by a first order Taylor series.

$$\hat{f}(X, Y) = f(\mu_X, \mu_Y) + f'_X(\mu_X, \mu_Y)(X - \mu_X) + f'_Y(\mu_X, \mu_Y)(Y - \mu_Y)$$

- The variance can be approximated.

$$\begin{aligned} \text{Var}[f(X, Y)] \approx\; & (f'_X(\mu_X, \mu_Y))^2 \sigma_X^2 + (f'_Y(\mu_X, \mu_Y))^2 \sigma_Y^2 \\ & + 2 f'_X(\mu_X, \mu_Y) * f'_Y(\mu_X, \mu_Y) * \sigma_{X,Y} \end{aligned} \tag{2}$$

Estimation of variance of $\hat{S}_M$

$\hat{S}_M$ is a function of 4 random variables, recoveries at Antioch and Chipps Island from releases from Durham Ferry, and recoveries at Antioch and Chipps Island from releases at Mossdale.

$$\begin{aligned} \hat{S}_M &= f(Y_{DA}, Y_{DC}, Y_{MA}, Y_{MC}) && (3) \\ &= \frac{(Y_{DA} + Y_{DC})/R_D}{(Y_{MA} + Y_{MC})/R_M} && (4) \end{aligned}$$

The mean, variances, and covariances of the 4 random variables are based on two trinomial distributions. These means, variances, and covariances must be estimated.

$$\begin{aligned} \hat{\mu}_{Y_{DA}} &= Y_{DA} && (5) \\ \hat{\mu}_{Y_{DC}} &= Y_{DC} && (6) \\ \hat{\mu}_{Y_{MA}} &= Y_{MA} && (7) \\ \hat{\mu}_{Y_{MC}} &= Y_{MC} && (8) \end{aligned}$$

$$\begin{aligned} \widehat{Var}(Y_{DA}) &= R_D \frac{Y_{DA}}{R_D}(1 - \frac{Y_{DA}}{R_D}) = Y_{DA}(1 - \frac{Y_{DA}}{R_D}) && (9) \\ \widehat{Var}(Y_{DC}) &= Y_{DC}(1 - \frac{Y_{DC}}{R_D}) && (10) \\ \widehat{Var}(Y_{MA}) &= Y_{MA}(1 - \frac{Y_{MA}}{R_M}) && (11) \\ \widehat{Var}(Y_{MC}) &= Y_{MC}(1 - \frac{Y_{MC}}{R_M}) && (12) \\ \widehat{Cov}(Y_{DA}, Y_{DC}) &= -R_D \frac{Y_{DA}}{R_D} \frac{Y_{DC}}{R_D} && (13) \\ \widehat{Cov}(Y_{MA}, Y_{MC}) &= -R_M \frac{Y_{MA}}{R_M} \frac{Y_{MC}}{R_M} && (14) \end{aligned}$$

The other covariances are zero.

$$\begin{aligned} f'_{Y_{DA}} &\propto \frac{1}{Y_{MA} + Y_{MC}} && (15) \\ f'_{Y_{DC}} &\propto \frac{1}{Y_{MA} + Y_{MC}} && (16) \\ f'_{Y_{MA}} &\propto \frac{-(Y_{DA} + Y_{DC})}{(Y_{MA} + Y_{MC})^2} && (17) \\ f'_{Y_{MC}} &\approx \frac{-(Y_{DA} + Y_{DC})}{(Y_{MA} + Y_{MC})^2} && (18) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\hat{S}_M) &\approx \left(\frac{R_M}{R_D}\right)^2 \times [(f')^2_{Y_{DA}} \widehat{Var}(Y_{DA}) + (f')^2_{Y_{DC}} \widehat{Var}(Y_{DC}) \\ &+ (f')^2_{Y_{MA}} \widehat{Var}(Y_{MA}) + (f')^2_{Y_{MC}} \widehat{Var}(Y_{MC}) \\ &+ 2 f'_{Y_{DA}} f'_{Y_{DC}} \widehat{Cov}(Y_{DA}, Y_{DC}) + 2 f'_{Y_{MA}} f'_{Y_{MC}} \widehat{Cov}(Y_{MA}, Y_{MC})] \end{aligned}$$